Preface

This book is intended for a wide readership of mathematicians and physicists, advanced graduate level students, masters and higher degrees in mathematics and mathematical physics. The presentation is clear and well organized, and many examples and problems are provided throughout the text. The contents comprise four chapters devoted to the following topics in detail.

Chapter 1 deals with the study of some properties of one-parameter groups of diffeomorphisms or flow, Lie derivative, interior product and Cartan's formula. We review some interesting properties and operations on differential forms, with detailed proofs.

Chapter 2 is devoted to the celebrated Poincaré lemma or Volterra theorem about differential forms. This lemma is of utmost importance, both in theory and in practice. It tells us that every differential form which is closed is locally exact. In other words, on a contractible manifold, all closed forms are exact. The aim of this chapter is to present some direct proofs of this result and explore some of its consequences. We will give several formulations of the Poincaré lemma. Also, some examples and questions closely related to this lemma will be discussed.

In Chapter 3, we will define the notion of a sheaf and a presheaf and show some of their basic properties. Sheaves are useful in situations when the passage of data from local to global is necessary. Sheaf cohomology has become absolutely essential in modern algebraic geometry, as well as certain areas of topology, differential geometry and complex geometry. One of the useful uses of cohomology is that if two spaces have different cohomology groups, then they cannot be homeomorphic; in other words, two homeomorphic spaces have isomorphic cohomology groups. We study Čech cohomology; this is based on the intersection properties of open covers of a topological space. It is a mathematical tool for assembling local algebraic data into global structures. The idea is that if we have information about the open sets that make up a space and how these sets are glued together, we can deduce global properties of the space from local data. We also study Leray theorem, which states that the cohomology groups of sheaf on a topological space can be computed by means of an arbitrary acyclic covering without using the direct limit procedure.

Chapter 4 is devoted to some connections with Čech–De Rham–Dolbeault cohomologies, $\overline{\partial}$–Poincaré lemma or Dolbeault–Grothendieck lemma. We begin discussing a few facts about the De Rham cohomology. This is an invaluable tool for the study of differential forms on smooth manifolds and is an important topological invariant. We explore the Mayer–Vietoris sequence which is a powerful instrument for computing the cohomology of the union of two open sets. As an application, we will compute the cohomology of the sphere, using the Mayer–Vietoris sequence and some more elegant machinery. Afterward, we introduce the Künneth formula and show how it can be used to calculate the cohomology group of the torus. The rest is devoted to the study of $\overline{\partial}$–Poincaré lemma or the Dolbeault–Grothendieck lemma and its consequences. This is a fundamental result of the Dolbeault cohomology; it is the analogue of Poincaré lemma for the De Rham theory (polydiscs have no higher cohomology for the sheaf of (p, q)-forms). We begin with the generalized Cauchy integral formula for smooth functions; and using this, we will prove the $\overline{\partial}$–Poincaré lemma in one variable. Next, we study the $\overline{\partial}$–Poincaré lemma or Grothendieck Poincaré lemma. We are also interested in the study of the Dolbeault theorem, establishing the isomorphism between Dolbeault and Čech cohomology. As an application of the Dolbeault theorem, we prove a special case of the Leray theorem for the sheaf of holomorphic functions. The chapter ends with a few results related to connections, curvature and first Chern class of line bundles.

The main text is enriched by numerous concrete examples, exercises and their solutions. This book covers a wealth of very important material in a concise, but nevertheless very instructive manner; and as such, it may serve as an excellent guide to further, more advanced and detailed reading in this fundamental area of mathematics.

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